Strain Localization Due to Structural Softening During Pressure Sensitive Rate Independent Yielding

LAETITIA LE POURHIET

Key-words. - plasticity ; linear stability analysis ; geodynamic modeling ; Mohr-Coulomb plasticity

Abstract. - Après avoir obtenu la solution analytique complète décrivant la courbe d’adoucissement du modèle élasto-plastique non associé de Mohr-Coulomb (MC-model), cet article démontre que cette rhéologie aboutit après une phase d’adoucissement à un changement critique fini ce qui permet de résoudre le problème de la chute de contrainte en l’abordant comme un problème ne dépendant que des conditions aux limites. Le MC-model produit une chute de contrainte qui adimensionnée, ne dépend que de trois paramètres : α_0, l’orientation de la bande de cisaillement par rapport la contrainte principale mineure en dehors de la bande ; φ, l’angle de friction interne au pic de résistance and ψ l’angle de dilatation plastique de la bande. Lorsque la réduction de la résistance de la bande avec la déformation est maximum pour ce modèle, la chute de contrainte est du même ordre de grandeur que la pression de confinement \( p_0 \). Dans ce régime le plus faible, la chute de friction effective de la bande de cisaillement s’effectue depuis une friction au pic de \( \mu_{\text{ini}} = 0.85 \) jusqu’à une friction résiduelle de \( \mu_{\text{ss}} = 0.64 \). Contrairement aux modèles ne prenant pas en compte l’épaisseur des bandes de cisaillement, la direction la plus faible ne correspond pas exactement l’orientation de Coulomb. Elle en dévie systématiquement et cette déviation augmente avec l’angle de friction interne du matériau. Cet article apporte aussi une quantification semi-analytique de la quantité de déformation prédite par le modèle pour parfaire la phase d’adoucissement. Cette quantité de déformation s’avère être très faible, de l’ordre de 7-8% pour des pressions de confinement et des angles de friction caractéristiques de la croûte supérieure terrestre.

Localisation de la Déformation Causée par l’Adoucissement Structural durant la Ruine de Matériau dont la Rhéologie Dépend de la Pression mais pas du Taux de Déformation

Mots-Clés. - plasticité ; analyse de stabilité linéaire ; modélisation géodynamique ; plasticité de Mohr-Coulomb

Résumé - Après avoir obtenu la solution analytique complète décrivant la courbe d’adoucissement du modèle élasto-plastique non associé de Mohr-Coulomb (MC-model), cet article démontre que cette rhéologie aboutit après une phase d’adoucissement à un changement critique fini ce qui permet de résoudre le problème de la chute de contrainte en l’abordant comme un problème ne dépendant que des conditions aux limites. Le MC-model produit une chute de contrainte qui adimensionnée, ne dépend que de trois paramètres : α_0, l’orientation de la bande de cisaillement par rapport la contrainte principale mineure en dehors de la bande ; φ, l’angle de friction interne au pic de résistance and ψ l’angle de dilatation plastique de la bande. Lorsque la réduction de la résistance de la bande avec la déformation est maximum pour ce modèle, la chute de contrainte est du même ordre de grandeur que la pression de confinement \( p_0 \). Dans ce régime le plus faible, la chute de friction effective de la bande de cisaillement s’effectue depuis une friction au pic de \( \mu_{\text{ini}} = 0.85 \) jusqu’à une friction résiduelle de \( \mu_{\text{ss}} = 0.64 \). Contrairement aux modèles ne prenant pas en compte l’épaisseur des bandes de cisaillement, la direction la plus faible ne correspond pas exactement l’orientation de Coulomb. Elle en dévie systématiquement et cette déviation augmente avec l’angle de friction interne du matériau. Cet article apporte aussi une quantification semi-analytique de la quantité de déformation prédite par le modèle pour parfaire la phase d’adoucissement. Cette quantité de déformation s’avère être très faible, de l’ordre de 7-8% pour des pressions de confinement et des angles de friction caractéristiques de la croûte supérieure terrestre.

1. Université Pierre et Marie Curie, UMR 7193, ISTEP, F-75005, Paris, France
2. Centre National de la Recherche Scientifique, UMR 7193, ISTEP, F-75005, Paris, France

Auteur correspondant. Tel : +033144275883 ; fax : +033144275883 ; Email : laetitia.le_pourhiet@upmc.fr.

Manuscript received on December 20, 2011 ; accepted on April 4, 2012.
Faults and mylonites are natural examples of localized shear strain. At a first glance, both display the same overall structure, i.e. a relatively thin zone of intense shear which separates relatively undeformed blocks. Field observations [Tchalenko and Ambroseys, 1970] and experiments [Tchalenko, 1970] show that localized structure results from the coalescence of en-echelon cracks, veins and smaller faults or strands. This apparent scaling of the structures is an argument in favor of their self-localization. In the laboratory, all rocks exhibit similar properties at ambient temperature and for a variety of triaxial stress orientations:

1. the failure/yield strength is pressure sensitive [Coulomb, 1776];
2. the bulk strain is very small compare to the shear strain within the shear bands;
3. the sensitivity of strength to pressure is constant for all kinds of pre-cut rocks [Byerlee, 1978];
4. localization starts with a phase of plastic hardening, during which diffused micro-cracking occurs [Lockner et al., 1991];
5. even in real triaxial tests, shear band planes strike along the intermediate principal stress axis e.g. [Mogi, 1971]

Within plasticity theory, the first three properties infer that rocks follow a non associated plastic flow rule which necessarily implies that at least two independent rheological coefficients, i.e. one for friction and one for dilatation, are needed in order to characterize the pressure sensitivity of the plastic flow in rocks. The third property implies the generality of the observed behavior. The fourth property states that localization is syn-peak of strength instability which occurs within a phase of mobilization of friction. The last property infers that shear bands are 2D plane strain objects by nature. Hence, Mohr-Coulomb non associated plasticity (MC-model), which is a 2D plane strain flow rule, is a more accurate representation of the results of common tri-axial test than the Drucker-Prager yield criterion [Drucker and Prager, 1952]. Although the analysis in restricted to 2D plane strain approximation in this paper, it should be noted that the 2D nature of the MC-model implies the existence of vertices within the yield criterion and flow rule in 3D. Those vertices allow for stress to concentrate [Rice and Rudnicki, 1980], and therefore, favor self localization of strain over smoother pressure dependent yield criteria such as Lade and Duncan [1975] or Drucker and Prager [1952]. Vermeer [1990] proposed a simple physical model aimed at reproducing the stress-strain response of rocks to bi-axial loading and simple shear tests obtained in the laboratory. Using the MC-model and constant coefficients of friction and dilatation, the resulting stress-strain curves showed a non linear strain softening behavior and the stress drop obtained were similar to those obtained experimentally. However, Vermeer[1990] contribution considered a material with a Poisson ratio, ν = 0, and concentrated his effort at computing the stress drop a shear band can achieve given its orientation. Since we are interested in the rate of structural softening, the effect of the elastic compressibility has been reintroduced in Vermeer [1990] formulation. This allows us to establish the critical finite strain needed for an elasto-plastic shear band to reach the maximum stress drop that its orientation allows to achieve. Multiplied by the thickness of faults, this critical strain corresponds to the critical slip L in the slip weakening model [Kanamori et Brodski, 2004 ]. In section Model, we introduce the concept of structural softening with a comprehensive qualitative description which utilizes Mohr circles. Then, we formulate the problem quantitatively in a linearized form which we solve analytically in terms of incremental stress and strains. We then demonstrate that all shear bands which form within this model are reaching a steady state strength, or a limit load asymptotically. This limit load is quantified before quantifying the finite strain involved during the softening phase by the means of a semi-analytical parametric study. The paper finally proposes a good empirical approximation for this finite strain and discusses how to use these results to solve some numerical issues encountered in the field of geodynamics, and the validity of the model for rocks at long and short timescale.

Model

A continuum mechanics approach

The paper considers a point-wise, 2D plane strain formulation that aims at modeling the neoformation of a single shear band in a elasto-plastic media by the means of classical post-bifurcation analysis e.g. [Hill and Hutchinson, 1975; De Borst, 1987; Vardoulakis et al., 1978; Vermeer, 1990]. We will assume that the material is a continuum. Quantities defined inside and outside the shear band will be denoted by the superscript in and out respectively. The boundary conditions of the model reflect a bi-axial compression test or the neoformation of an inverse fault (Fig. 1A). Considering both jacketed laboratory experiments and Anderson (1951) mechanics, the least principal stress acting in the far field of the shear band (i.e. \( \sigma_{3}^{\text{out}} \) here) is fully prescribed and corresponds to the constant confining pressure (\( p_{0} \)). For
crustal scale faulting application, the confining pressure is the vertical stress so that \( p_0 = \sigma_z = \int_0^z \rho(z) \, g \, dz \) where \( \rho(z) \) is the density and \( g \) is the Earth’s gravity acceleration and \( z \) is depth. By fixing the orientation of \( \sigma_z^{in} \), the orientation of \( \sigma_z^{out} \) is fixed but not its amplitude which depends on the effective frictional properties of the shear band.

The shear band orientation defines the \( \xi \)-axis of a local orthonormal reference frame \( \xi - \eta \) in which the state of stress acting on the shear band is best described in the Mohr space (see Fig. 1C). The strains considered in this study are small enough to neglect the passive rotation of the shear band with strain. It results that the shear bands, and therefore the \( \xi \)-axis, forms a constant angle \( \theta^{out} = \frac{\pi}{4} + \frac{\phi_{\sigma}}{2} \) with \( \sigma_\xi^{in} \). In the \( \xi - \eta \) reference frame, following a continuum approach, the compatibility condition

\[
\varepsilon^{in}_{\xi \xi} = \varepsilon^{out}_{\xi \xi}, \quad (1)
\]

must be satisfied to ensure that no voids form at the interface of the shear band while the continuity condition on shear \( \tau \) and normal stress \( \sigma_n \) acting on the fault plane

\[
\sigma_n = \sigma_{\eta\eta}^{in} = \sigma_{\eta\eta}^{out}, \quad (2)
\]

\[
\tau = \sigma_{\eta\eta}^{in} = \sigma_{\eta\eta}^{out}, \quad (3)
\]

according to Amonton’s Law. In our point-wise approach, only \( \sigma_{\xi\eta}^{in} \) and \( \varepsilon_{\eta\eta}^{out} \) are allowed to be discontinuous across the shear band boundary [Vermeer, 1990]. This possible discontinuity implies that the principal axis of stress may rotate within the shear band. In the Mohr space it results that the state of stress must be described by two circles, a grey one within the band and black one outside the band. The continuity condition implies that the two circles must intersect at the shear band plane. In the derivation, the constant angle \( \alpha_0 \) and variable angle \( \beta \) have been introduced to replace the constant \( \theta^{out} = \frac{\pi}{4} + \frac{\phi_{\sigma}}{2} \) and the variable \( \theta^{in} = \frac{\beta}{2} + \frac{\phi_{\eta}}{2} \) in order to shorten and simplify the derivations, these angles appear on Fig. 1B. Finally, in this approach, it is important to note that only the Mohr circle associated to the shear band, i.e the black one, must always be tangent to the MC yield criterion which is represented by a line in the Mohr space following

\[
\tau = -\tan \phi \sigma_n + \alpha_0, \quad (4)
\]

where \( Co \) is the cohesion and \( \phi \) is the peak or intrinsic friction angle of the material. In this contribution we prefer to express the yield criteria using the yield function

\[
\mathcal{F} = R + \sin \phi C - \cos \phi, \quad (5)
\]

which describes the distance between a Mohr circle and standard failure line Eq. 4 in the Mohr space as a function of the radius of the circle

\[
R = \frac{\sigma_1 - \sigma_3}{2}, \quad (6)
\]

and the center of circle

\[
C = \frac{\sigma_1 + \sigma_3}{2}. \quad (7)
\]

When \( \mathcal{F} < 0 \) the medium is elastic. When \( \mathcal{F} = 0 \) the medium is elasto-plastic and during yielding, \( \mathcal{F} \) cannot exceed 0. The state of stress in the yielding shear band may be described by the radius

\[
\tau^* = \frac{1}{2} \sqrt{\left(\sigma_{\xi\xi} - \sigma_{\eta\eta}\right)^2 + 4\sigma_{\xi\eta}^2} \quad (8)
\]

and the center

\[
\sigma^* = \frac{1}{2} \left(\sigma_{\xi\xi} + \sigma_{\eta\eta}\right). \quad (9)
\]

of the black Mohr circle (Fig. 1C), and consequently,

\[
\mathcal{F} = \tau^* + \sin \phi \sigma^* - C \cos \phi = 0, \quad (10)
\]

is always true in the MC-Model.

Structural softening

The term of structural softening/hardening is imported from engineering/architecture and refers to the geometric forcing that can lead to reduction/increase of the effective strength of a material without changing the intrinsic properties of this material. It is not specific to the non associated flow rule and in geosciences it has been shown that large strain folding and buckling leads to structural softening of the strong layers [Schmalholz et al., 2005]. Structural hardening related to the play of inherited faults using non associated plasticity is described in details in Lecomte et al. [2011]. In this article, we focus on the neoformation of shear bands, also referred as shear localization, in material obeying non associated plastic flow rule. Before entering into a more quantitative approach, it is important to define geometrically the effect of non associated elasto-plastic rheology on the relative orientation of stress and flow during plastic yielding.

During the whole evolution, five rules apply (see Fig. 1) :

1. the shear band forms an angle of \( \alpha_0 / 2 \) with the maximum shear stress outside the band ;
2. the shear band forms an angle \( \beta / 2 \) with the maximum shear stress inside the band ;
3. the plastic flow direction (white arrow) forms an angle \( \psi \) with the maximum shear stress inside the band (Fig. 1B) ;
4. the state of stress within the band is always at yield, i.e. the black circle is tangent to the yield surface ;
5. the continuity condition ensures that the shear stress \( \tau = \sigma_{\xi\eta} \) and the normal stress \( \sigma_n = \sigma_{\eta\eta} \) acting on the band are the same inside and outside, i.e. the two circles must intersect at the orientation of the band and define the effective friction coefficient,

\[
\mu_{\eta\eta} = -\frac{\sigma_{\xi\eta}}{\sigma_{\eta\eta}} \quad (11)
\]

At the initial stage, the state of stress is exactly the same within and outside the shear band. As a result, \( \beta = \alpha_0 \) and the state of stress is represented by a single Mohr circle which tangents the MC yield criteria, Eq. 10, represented by a line in the Mohr space. In this situation, unless \( \alpha_0 = \psi \), the dilatation angle, the direction of the plastic flow within the shear band is not parallel to the direction of the shear band (Fig. 2A).

This initial discrepancy between total strain (\( \gamma \)) and plastic strain (white arrow, Fig. 2A) must be accommodated elastically within the shear band. The incremental elastic strain tensor is therefore not parallel to the cumulated elastic strain tensor, causing a rotation of the principal axes of the cumulated elastic strain tensor. As a result, the internal principal stress axes (black cross, referred by \( \theta \)) rotates implying that \( \beta \neq 0 \) during that phase of yielding (Fig. 2B).
As yielding continues and shear strain $\gamma$ increases, the principal stress axes reach an orientation such that $\beta(\gamma_c) = \beta_{ss} = \psi$. After that critical shear strain $\gamma_c$, the plastic flow becomes parallel to the shear band (Fig. 2C). Once the $\beta_{ss}$ orientation is reached, no elastic strain occurs within the shear band. The stress tensor stops rotating ($\dot{\beta} = 0$) and the stress does not change anymore ($\dot{\tau} = 0$). Hence, this state is denoted by $(\beta_{ss})$ for steady state. After that critical strain, the shear band behaves as an effectively perfect plastic body of friction $\mu_{ss}$ (see Section ), which can be read graphically in the Mohr diagram as the slope of the gray line.

For the initial condition prescribed in the example of the Fig. 2A, the external Mohr circle goes away from the yield surface as its radius decreases Fig. 2B,C. This means that as the shear bands yield and deforms plastically, respecting the consistency condition i.e.

$$\mathcal{F}_{in} = 0 \cup \mathcal{F}_{in} = 0.$$  \hspace{1cm} (12)

the external part is unloaded elastically, i.e.

$$\mathcal{F}_{out} < 0.$$  \hspace{1cm} (13)

These two relations are the two necessary and sufficient conditions for strain localization to occurs with that model [Vardoulakis et al., 1978; Vermeer, 1990]. During the phase of strain localization, the material intrinsic properties have not changed. The neoformation of a shear band with that model can therefore be qualified as the result of structural softening. The shear bands which orientation leads to the elastic unloading of the surrounding material will now be named as well oriented shear bands for neoformation. However, it should be noted that the rotation of the stress tensor with shear strain may also cause an increase of the strength of the shear band with strain. This hardening case is well developed and applied to inherited low angle normal faults in Lecomte et al. [2011]. Here, we concentrate on the neoformation of shear bands and on the critical shear strain needed to decrease the shear strength to its steady state value.
Quantifying the strength of the system

This part of the paper is devoted to establishing a relationship between the orientation of stress ($\mathcal{F}$) within a shear band and the ratio of maximum principal stress $\sigma^\text{out}$ to the confining pressure $p_0$, which is denoted $\mathcal{F}$ for the general case described on Fig. 2C. $\mathcal{S}$ is used here to measure the strength of the system composed of the shear band embedded in the elastic media because $(\mathcal{F} - 1)p_0$ is the maximum deviatoric stress of the system, which corresponds to both the radius of the Mohr circle outside and the abscissa of the plastic yield strength in the yield strength envelope [Brace and Kohlsted, 1980]. The key to establishing such a relationship is the continuity condition. Hence, we seek to express the effective friction using external and internal stress components alternatively. The constant orientation $\alpha_0$ of the shear band within the external stress field, allows expressing all the components of the tensor ($\sigma^\text{out}$) through a linear relationship of $\mathcal{F}$ with:

$$\sigma^\text{out} = \begin{bmatrix} \sigma^\text{out}_{\xi} \\ \sigma^\text{out}_{\eta} \\ \sigma^\text{out}_{\xi\eta} \end{bmatrix} = \frac{1}{2} p_0 \begin{bmatrix} (\mathcal{F} + 1) + (\mathcal{F} - 1)\sin \alpha_0 \\ (\mathcal{F} + 1) - (\mathcal{F} - 1)\sin \alpha_0 \\ -(\mathcal{F} - 1)\cos \alpha_0 \end{bmatrix},$$

(14)

Taking the ratio of $\sigma^\text{out}_{\xi\eta}$ to $\sigma^\text{out}_{\eta\eta}$ as in Eq. 11, reaches an expression that relates the effective friction $\mu_e$, the strength of the system $\mathcal{S}$ and the orientation of the band $\alpha_0$ following

$$\mathcal{F} = \frac{\cos \alpha_0 + \mu_e (\sin \alpha_0 + 1)}{\cos \alpha_0 - \mu_e (1 - \sin \alpha_0)}$$

(15)

It is now possible to insert the expression of $\mu_e(\mathcal{F})$ in the previous expression to compute the strength of the system as a function of the orientation of the internal stress relative to the shear band ($\mathcal{F}$). This relation,

$$\mu_e = \frac{\sin \phi \cos \beta \sin \beta}{1 - \sin \phi \sin \beta},$$

(16)

is found first by expressing the shear and normal stress as a function of the invariants

$$\begin{cases} \tau = \sigma^\text{int}_{\xi\eta} = \tau^* \cos \beta \\ \sigma_n = \sigma^\text{int}_{\eta\eta} = \sigma^* + \tau^* \sin \beta \end{cases},$$

(17)

and second by assuming that since the material inside the fault has no cohesion and is yielding following eq. 10, one finds

$$\sigma^* = -\frac{\tau^*}{\sin \phi}.$$ (18)

Rheology

In order to compute the evolution of the system, it is necessary to relate the stress to strain and therefore to introduce the necessary rheological relationships in this mechanical system that allows formulating the incremental problem. The derivations of the paper are only meant to describe the onset of shear banding and we will show in a later section on characterization of transient softening that 15% strain is an extreme value of $\gamma$ for rocks in the Earth’s tectosphere conditions and that 5 to 7% is a more relevant one. Hence, and for the sake of simplicity, we use the Cauchy stress tensor and drop the Jaumann co-rotational correction terms from the derivation. In the local $\xi - \eta$ reference frame, the stress, stress rate and strain rate tensor are written using vector notation so that the constitutive equations may take the form of a rigidity matrix $R$ rather than a fourth order tensor, which we denote via

$$\hat{\sigma}_i = R_{ij} \hat{e}_j.$$ (19)

In order to maintain objectivity of the constitutive relationship within this compact notation, the engineering strain rate formulation is used and hence the strain rate is

$$\dot{\varepsilon} = \begin{bmatrix} \dot{\varepsilon}_{\xi\xi} \\ \dot{\varepsilon}_{\eta\eta} \\ \dot{\varepsilon}_{\xi\eta} \end{bmatrix},$$

(20)

where $\gamma = 2\dot{\varepsilon}_{\xi\eta}$. Outside the shear band the material is elastic and follows Hooke’s law for an isotropic material, hence only two material parameters are required to describe the constitutive behavior. Here, we have chosen the shear modulus $G$ and the Poisson ration $\nu$. In our vector/matrix notation the rigidity matrix for elastic behavior, $D$ is given by

$$D = \begin{bmatrix} 2(1-\nu) & \nu G & 0 \\ \nu G & 2(1-\nu) & 0 \\ 0 & 0 & G \end{bmatrix}$$

(21)

together and the elastic flow rule is

$$\dot{\varepsilon}^e = D^{-1} \dot{\sigma},$$

(22)

with

$$\dot{\sigma}^\text{out} = \frac{\mathcal{F} p_0}{2} \begin{bmatrix} 1 + \sin \alpha_0 \\ 1 - \sin \alpha_0 \\ -\cos \alpha_0 \end{bmatrix}.$$ (23)

Within the shear band the material is elasto-plastic, and the flow rule assumes a Maxwell summation of individual strain rate components,

$$\dot{\varepsilon}^p = \dot{\varepsilon}^e + \hat{\varepsilon}^p.$$ (24)

The plastic flow rule is proportional to the plastic potential $\mathcal{Q}$, differentiated with respect to stress

$$\dot{\varepsilon}^p = \lambda \frac{\partial \mathcal{Q}}{\partial \sigma},$$

(25)

where the proportionally constant $\lambda$ is referred to as the plastic multiplier. The material behaves purely elastically, $\lambda \equiv 0$, when the state of stress does not respect the consistency condition written in Eq. (12). The plastic potential $\mathcal{Q}$ is defined following Vermeer and Deborst [1984] by

$$\mathcal{Q} = \tau^* + \sin \psi \sigma^*.$$ (26)

We note that Eq. (26), resembles the yield criterion $\mathcal{F}$ from Eq. (5), however the bulk strain sensitivity to pressure is expressed in terms of a dilatancy angle $\psi$, which is defined as the ratio

$$\tan \psi = \frac{\dot{\varepsilon}_{\xi\eta}^p}{\dot{\varepsilon}_{\xi\eta}^\text{out}}.$$ (27)

Using trigonometric relations found from Mohr’s circle on Fig. 2, one finds $\sin \beta = \frac{\sigma_{\eta\eta} - \sigma_{\xi\xi}}{2\sigma_{\xi\eta}}$ and $\cos \beta = \frac{\sigma_{\xi\xi} + \sigma_{\eta\eta}}{2}$; the derivative of the plastic potential with respect to stress may be expressed as a function of $\beta$

$$\frac{\partial \mathcal{Q}}{\partial \sigma} = \begin{bmatrix} \sin \psi - \sin \beta \\ \sin \psi + \sin \beta \end{bmatrix}^T.$$ (28)

"Bull. Soc. gol. Fr., 2012, no 9"
**Analytical formulation of the incremental problem**

Outside the shear band, the material behaves as an homogeneous isotropic elastic body. By combining Eq. 23 and Eq. 22 one therefore obtains

\[ \epsilon^{\text{out}} = D^{-1} \sigma^{\text{out}} = \frac{\rho_0}{4G} \begin{bmatrix} 1 + \sin \alpha_0 - 2\nu & 1 - \sin \alpha_0 - 2\nu \\ 1 - \sin \alpha_0 - 2\nu & -1 \end{bmatrix} \]. \quad (29)

Hence, the measurable variable \( \mathcal{S} \), its time derivative \( \dot{\mathcal{S}} \) and the four parameters : \( \rho_0 \), \( \alpha_0 \), \( G \), \( \nu \) are sufficient to fully describe the evolution of the elastic block. Inside the band, continuity (Eq. (2)) and compatibility (Eq. (1)) prescribe three out of the six components of the stress rate and strain rate tensors so that \( \dot{\sigma}^{\text{int}}_{\eta \xi} = \dot{\sigma}^{\text{out}}_{\eta \xi} \), \( \dot{\sigma}^{\text{int}}_{\eta \eta} = \dot{\sigma}^{\text{out}}_{\eta \eta} \) and \( \dot{\epsilon}^{\text{int}}_{\eta \eta} = \dot{\epsilon}^{\text{out}}_{\eta \eta} \). These three components therefore depend linearly on the strength rate, \( \dot{\mathcal{S}} \). For the last three unknown increments (\( \dot{\epsilon}^{\text{int}}_{\eta \eta}, \dot{\epsilon}^{\text{int}}_{\eta \xi} \) and \( \dot{\sigma}^{\text{int}}_{\eta \xi} \)), it is necessary to add three additional linear equations to describe the elasto-plastic behavior of the shear band. Using Eq. (24), we obtain

\[ D^{-1} \begin{bmatrix} \dot{\sigma}^{\text{int}}_{\eta \eta} \\ \dot{\sigma}^{\text{int}}_{\eta \xi} \\ \dot{\sigma}^{\text{int}}_{\xi \xi} \end{bmatrix} + \lambda \frac{\partial \mathcal{Q}}{\partial \sigma} \begin{bmatrix} \dot{\epsilon}^{\text{int}}_{\eta \eta} \\ \dot{\epsilon}^{\text{int}}_{\eta \xi} \\ \dot{\epsilon}^{\text{int}}_{\xi \xi} \end{bmatrix} = 0. \quad (30) \]

In establishing that the shear band obeys an elasto-plastic flow rule, \( \lambda \) has been introduced as a fourth unknown. Assuming that the shear band will continually yield, the three rheological equations (Eq. 30) can be solved simultaneously in conjunction with the following consistency condition (Eq. 12)

\[ \dot{\mathcal{S}} = \frac{\partial \mathcal{F}}{\partial \sigma} \dot{\sigma}^{\text{int}} = \begin{bmatrix} \frac{\sin \phi - \sin \beta}{2} & \frac{\sin \phi + \sin \beta}{2} & \cos \beta \end{bmatrix} \begin{bmatrix} \dot{\sigma}^{\text{int}}_{\eta \eta} \\ \dot{\sigma}^{\text{int}}_{\eta \xi} \\ \dot{\sigma}^{\text{int}}_{\xi \xi} \end{bmatrix} = 0. \quad (31) \]

In order to obtain a solution. Note that the similitude existing between the expressions of \( \mathcal{F} \) and \( \mathcal{Q} \) allows us to obtain \( \frac{\partial \mathcal{F}}{\partial \sigma} \) by replacing \( \psi \) by \( \phi \) in the expression of \( \frac{\partial \mathcal{Q}}{\partial \sigma} \) (Eq. 28). Laboratory measurements preclude the use of \( \mathcal{F} \) as a free parameter. Therefore, we used Eqs. (30) and (31), to express our four unknowns \( x = [\lambda, \dot{\epsilon}^{\text{int}}_{\eta \eta}, \dot{\epsilon}^{\text{int}}_{\eta \xi}, \dot{\sigma}^{\text{int}}_{\xi \xi}] \) in terms of \( \mathcal{S} \), write the system in the matrix form

\[ Ax = b, \quad (32) \]

where the 4x4 matrix \( A \) and the vector \( b \) are fully spelled out in \( A \) with the full solution of this linear system of equation. However, we would like to point out here that this system does not have a unique solution whenever \( \beta = \phi \) or \( \beta = \psi \) since

\[ \det A = -\frac{1}{4}(\sin \phi - \sin \beta)(\sin \beta - \sin \psi). \quad (33) \]

Using the solution of Eq. 32 in the interval \( [\psi, \phi] \) allows to express the strength rate of change

\[ \dot{\mathcal{S}} = -\frac{2G}{\rho_0} \Gamma \dot{\epsilon}^{\text{int}}_{\eta \eta}, \quad (34) \]

as the product of the linear elastic update by a non-dimensional tangent modulus

\[ \Gamma = \frac{\det A}{Z \cos \beta (1 - \nu) + \det A \cos \alpha_0}. \quad (35) \]

Factor \( Z \) can be found in \( A \). Using this relation will allows us to integrate numerically the variation of strength with strain rate by updating the value of \( \beta \) at each increments (see B).

**Limit Analysis**

**Existence of a limit load**

Replacing \( \beta \) by \( \alpha_0 \) in Eq. (35), one can obtain an expression for the initial tangent modulus, which indicates that for shear banding to occur, i.e. for elastic unloading to take place outside the band, requires that

\[ \alpha_0 \in [\psi, \phi]. \quad (36) \]

Note that the singular cases, \( \beta = \phi \) and \( \beta = \psi \) have again
been excluded from the interval because there is a priori no unique solution for those two orientations as $\det A = 0$. At these two singular points, the strength rate of change outside the band is zero, i.e. $\dot{\gamma} = 0$. This implies that two components of the stress tensor are also constant inside the shear band, thus any small perturbation of $\sigma_{\eta \xi}^{\text{in}}$ will cause a rotation of the principal axis of the stress tensor. At those points, the expression of $\dot{\mu}$ found in Eq. 64 simplifies to

$$\sigma_{\eta \xi}^{\text{in}}|_{\gamma = 0} = \frac{G \lambda}{1 - \nu} (\sin \beta - \sin \psi).$$

(37)

Since $\lambda \propto \frac{\dot{\gamma}}{\det A}$ (Eq. 63)), its limit when $\beta \to \psi$ or $\beta \to \phi$ is an indeterminate form. From Eq. 65, one finds that $\lambda$ is indeed proportional to the shear strain within the shear band,

$$\lambda|_{\beta = \phi} = \frac{\gamma_i}{\cos \phi},$$
$$\lambda|_{\beta = \psi} = \frac{\gamma_i}{\cos \psi}$$

(38)

Consequently, if any kinematic forcing of the system exists, $\lambda$ cannot be zero. However, Eq. (37) shows that $\beta = \psi$ is a sufficient condition for both stress rate tensors of the model to cancel. On the one hand, when $\beta = \psi$, Fig. 2C the model reaches a constant (steady state) limit load and behaves similarly to a perfectly plastic body. On the other hand, when $\beta = \phi$, although the strength rate of the model is zero, and $\psi < \phi$ for typical rocks, the principal axis of the stress tensor rotates within the shear band. This rotation causes a modification of the effective friction, Eq. 16 which in turn modifies the strength Eq. 15. We now pose that for all well oriented shear bands, $\dot{\beta}$ tends to decrease from $\beta_{\text{ini}} \in [\psi, \phi]$ towards $\psi$ with increasing shear strain and thus, that all shear bands tend towards a steady state ($\beta = \psi$). To prove this, it is sufficient to ensure that $\beta \in [\psi, \phi]$ holds and that $\dot{\beta}$ is strictly negative. In this interval, we already know that $\dot{\gamma}$ is strictly negative. Hence, we write

$$\dot{\gamma} = \frac{\partial \gamma}{\partial \mu_{\text{ini}}} \frac{\partial \mu_{\text{ini}}}{\partial \beta}$$

(39)

and study the sign of the two partial derivatives. Beginning with $\partial \mu_{\text{ini}}/\partial \beta$, we find from Eq. 16 that

$$\frac{\partial \mu_{\text{ini}}}{\partial \beta} > 0 \quad \text{if} \quad \sin \beta < \sin \phi.$$  

(40)

Hence, provided $\dot{\gamma}$ is negative, i.e. $\beta \in [\psi, \phi]$, the first partial derivative is strictly positive. Now, using the same approach for $\partial \gamma/\partial \mu_{\text{ini}}$, we find from Eq. 15 that

$$\frac{\partial \gamma}{\partial \mu_{\text{ini}}} > 0 \quad \text{if} \quad \mu_{\text{ini}} < \frac{1 + \sin \phi}{\cos \phi}.$$  

(41)

Since $\mu_{\text{ini}}$ is an increasing function of $\beta$ on the interval $[\psi, \phi]$, it is straightforward to bound the effective friction on this interval:

$$\sin \phi \cos \psi \frac{1 - \sin \phi \sin \psi}{1 - \sin \phi \sin \psi} < \mu_{\text{ini}} < \tan \phi.$$  

(42)

Accounting for the physical constraint that $\phi < \pi/4$, we find that for well oriented shear bands,

$$\tan \phi < 1 \leq \frac{1 + \sin \phi_0}{\cos \phi_0}.$$  

(43)

As $\beta_{\text{ini}}$ must also lie in this interval for a shear band to form, it is evident that the principal stress axis of any newly formed shear band will progressively rotate towards an asymptotic orientation of $\beta_{\text{asympt}} = \psi$ where $\beta = 0$. This results means that, assuming a specific orientation, this model allows treating as a boundary value problem the stress drop achieved by a shear band from the peak strength at localization to the residual strength at steady state. In other words, using that model provides a physical interpretation to the variability of the stress drop without having to modify the friction of rocks.

**Stress drop as a boundary value problem**

Initially, we assume that the stress is at yield both within the shear band and the embedding elastic block. The Coulomb yield criteria can be rewritten as

$$\sigma_1 = \frac{1 + \sin \phi}{1 - \sin \phi} \sigma_3.$$  

(44)

Accounting for $\sigma_3 = p_0$, the initial strength is

$$\sigma_{\text{ini}} = \frac{1 + \sin \phi}{1 - \sin \phi}.$$  

(45)

As the initial state of stress is homogeneous, the orientation of $\sigma_1$ versus the shear band is the same both inside and outside. The initial value of $\beta$ is therefore $\phi_0$ in the model. Substituting this value of $\beta$ into Eq. (16), we get an expression of the initial effective friction,

$$\mu_{\text{ini}} = \frac{\sin \phi \cos \phi_0}{1 - \sin \phi \sin \phi_0}.$$  

(46)

At steady state, the effective friction of the shear band $\mu_{\text{ss}}$, is given by replacing $\beta$ by its steady state value $\psi$ in Eq. 16 to reach

$$\mu_{\text{ss}} = \frac{\sin \phi \cos \psi}{1 - \sin \phi \sin \psi}.$$  

(47)

Inserting $\mu_{\text{ss}}$ into Eq. (15), one obtains an expression for steady state strength,

$$\mathcal{F}_{\text{ss}} = \mu_{\text{ss}}(1 + \sin \phi_0) + \cos \phi_0,$$
$$\mu_{\text{ss}}(\sin \phi_0 - 1) + \cos \phi_0.$$  

(48)

From this, it is then possible to compute the total stress drop a shear band may achieve within the MC-model for a given orientation by subtracting the steady state strength to the initial strength obtained in Eq. 45 to yield

$$\mathcal{F}_{\text{tot}} \dot{\text{drop}} = \mathcal{F}_{\text{ini}} - \mathcal{F}_{\text{ss}}$$
$$= 2 \frac{\mu_{\text{ss}}(1 - \sin \phi \sin \phi_0) - \sin \phi \cos \phi_0}{(1 - \sin \phi)(\mu_{\text{ss}}(1 - \sin \phi_0) - \cos \phi_0)}.$$  

(49)

We can now use this expression to measure the impact of shear band orientation $\phi_0$ on stress drop and compare it to the impact of peak friction angle $\phi$ and dilatation angle $\psi$. Plotting the strength drop in a three parameters space for all shear band orientations, dilatation and friction angle, on Fig. 3, one sees that if the strength drop increases with friction angle, it also increases with shear band orientation. As a result, a fault, which orientation is far from Coulomb angle, produces a small strength drop even if the peak friction angle of the material in the fault is very high (35).
Specific shear band orientations

The two limiting shear band orientations for neoformation, i.e. \( \alpha_0 = \alpha_R = \psi \) and \( \alpha_0 = \alpha_C = \phi \), are known respectively as the Roscoe [Roscoe, 1970] and Coulomb [Coulomb, 1776] orientations. The Roscoe orientation does not lead to elastic unloading and is observed in samples close to the brittle plastic transition at high confining pressure [Besuelle, 2000] or in coarse grain samples [Vermeer, 1990]. The Coulomb orientation is observed at low confining pressure [Besuelle, 2000]. An other intermediate orientation \( \alpha_0 = \phi + \psi \) is often described in the laboratory [Arthur et al., 1977]. All the specific shear band orientations are represented on a Mohr circle in Fig. 3A. In Fig. 3B, the stress drop \( \Delta S_{\text{drop}} \) and the location of the special shear bands are represented in a three parameters space. We note that materials with a small peak friction, \( \phi \), such as mineralogical clays, cannot produce a large stress drop. Similarly, when \( \alpha_0 \) decreases below \( \alpha_4 \), the total stress drop is drastically reduced. In order to get more insight into the model and its relevance to describe laboratory experiments, we derive the specific orientations of shear bands that come out of the MC-model within the next two paragraphs. The first specific orientation \( \alpha_0 \), is named after Vermeer [1990]. It is the orientation of the shear band which minimizes the strength at steady state. We obtain it using Eq. (49) and solving \( \partial \Delta S_{\text{drop}}/\partial \alpha_0 = 0 \) to reach

\[
\frac{\partial \Delta S_{\text{drop}}}{\partial \alpha_0} = \cos \alpha_0 \mu_{ss} - \sin \alpha_0, \\
(50)
\]

from which we find using Eq. 47 that

\[
\begin{align*}
\tan \alpha_0 &= \tan \alpha_0 = \mu_{ss} = \frac{\sin \phi \cos \psi}{1 - \sin \phi \sin \psi} \\
\theta_V &= \frac{\pi}{4} + \frac{\alpha_0}{2}
\end{align*}
\]

(51)

It is worth noting that for non associated flow, in contrast to the usually assumed associated flow, the weakest mechanism is not exactly the Coulomb orientation \( \alpha_0 = \phi \), but this Vermeer orientation. Whilst these two orientations only differs by few degrees, the Vermeer orientation is always few degree smaller than Coulomb and better fits to results obtained numerically by Lemiale et al. [2008] in the rigid limit, or Kaus [2010] for both the rigid-plastic and elasto-plastic cases.

The second specific orientation corresponds to the shear band which maximizes the initial plastic strain rate and therefore the initial plastic dissipation. Using \( \beta = \alpha_0 \) in Eq. (63) we have the initial plastic multiplier

\[
\lambda_{\text{ini}} = \frac{\mathcal{F}_0 (1 - \sin \phi)(1 - \psi)}{4G \det A},
\]

(52)

The weakest initial shear band orientation is determined by computing

\[
\frac{d \lambda_{\text{ini}}}{d \alpha_0} = -\frac{\mathcal{F}_0 (1 - \sin \phi)(1 - \psi) \cos \alpha_0 (-2 \sin \alpha_0 + \sin \phi + \sin \psi)}{G (4 \det A)^2},
\]

(53)

It can be shown that Eq. (53) is identically zero in consideration of Eq. (36) when

\[
\begin{align*}
\sin \alpha_0 &= \sin \alpha_c = \frac{\sin \phi + \sin \psi}{2} \\
\theta_A &= \frac{\pi}{4} + \frac{\alpha_0}{2}
\end{align*}
\]

(54)

We name this specific orientation \( \alpha_4 \) after the Arthur orientation because it is found to deviate by less than 3 degrees from the intermediate orientation described in the laboratory. To conclude, it is worth noting here that the two weakest orientations found using the MC-model are within the error bar of the orientations which are very often observed in the laboratory, i.e. the Coulomb and Arthur orientations.

Characterization of the transient softening

The limit analysis has shown that a steady state exists for our model, from which one can compute a value for the steady state strength and steady state friction. In this section, we quantify the transient behavior, i.e. the strain softening part of the curve. In particular we focus on the amount of displacement/shear strain accumulated before the steady state is achieved.

Quantitative description of the stress-strain curves

The discussion focuses on the impact of the elasto-plastic properties of rock as well as that of shear band initial orientation \( \alpha_0 \) on the critical strain \( \gamma_c \) needed to reach steady state. Due to the asymptotic nature of the steady state, we need to define a finite criteria to characterize the end of the softening phase. We identify the system as being in a steady state when \( \beta \) equals the cut-off value \( \bar{\beta}_\text{c} \), defined by

\[
\bar{\beta}_\text{c} = \beta_{\text{cr}} + 0.02 (\beta_{\text{ini}} - \beta_{\text{cr}}).
\]

(55)

As a result, in the following, \( \gamma_c \) will correspond to the amount of shear strain \( \gamma_{\text{cr}}^{\text{ini}} \) when \( \beta = \bar{\beta}_\text{c} \). We will also describe the transient

FIG. 4 – Typical stress-strain curve obtained with the MC-Model. The non-dimensional strength \( \mathcal{F} \) diminishes non linearly with shear strain \( \gamma_{\xi \eta}^{\text{ini}} \) during the transient phase of softening. At each point the non-dimensional tangent modulus \( \Gamma \) and the non-dimensional secant modulus \( \Sigma \) may be defined. Steady state is reached when \( \Gamma = 0 \). This point defines the characteristic shear strain \( \gamma_c \).

Quantitative description of the stress-strain curves

The discussion focuses on the impact of the elasto-plastic properties of rock as well as that of shear band initial orientation \( \alpha_0 \) on the critical strain \( \gamma_c \) needed to reach steady state. Due to the asymptotic nature of the steady state, we need to define a finite criteria to characterize the end of the softening phase. We identify the system as being in a steady state when \( \beta \) equals the cut-off value \( \bar{\beta}_\text{c} \), defined by

\[
\bar{\beta}_\text{c} = \beta_{\text{cr}} + 0.02 (\beta_{\text{ini}} - \beta_{\text{cr}}).
\]

(55)

As a result, in the following, \( \gamma_c \) will correspond to the amount of shear strain \( \gamma_{\text{cr}}^{\text{ini}} \) when \( \beta = \bar{\beta}_\text{c} \). We will also describe the transient
shape of the stress strain curves (Fig. 4) by representing the non-dimensional strength $\mathcal{F}$, the non-dimensional tangent modulus $\Gamma$, the non-dimensional secant modulus $\Sigma$ with $\gamma_\text{elas}^p(t)$ during the transient phase of softening. Physically, the tangent modulus $\Gamma$ corresponds to the local slope, and hence is an indicator for the rate of softening. The secant modulus is

$$\Sigma = \frac{\mathcal{F}}{\gamma_\text{elas}^p(t)}.$$ (56)

This ratio of the current strength to the current shear strain can be considered as a proxy for the slope of a standard linear slip weakening model [Kanamori et Brodski 2004].

The shape of these curves will later be discussed for two aspects. The first one is purely technical and consists in fixing a time stepping which allow capturing the shape of these curves in numerical codes without committing large approximation errors that cause the divergence of non linear solution or the very slow convergence of non linear solvers. The second one consists in understanding if the MC-model within is apparent simplicity allows explaining some of the complexity of the architecture of natural fault zones. To that purpose, the paper provides the reader with typical values of stress drop, and characteristic strain or slip associated with this model.

**Effect of elastic parameters and confining pressure**

Fig. 5 illustrates the influence of the elastic parameters on the amount of strain needed to reach steady state and on the shape of the curve for a shear band oriented with Vermeer orientation ($\alpha_0 = \alpha_0$). In Fig. 5A, we show the variation of strength $\mathcal{F}$ with accumulated strain for a confining pressure which corresponds to a depth of 8 km, varying the values of the Poisson ratio. It is apparent that the MC-model predicts that the larger is the Poisson ratio the smaller is the critical strain $\gamma_c$. Fig. 5B represents the same kind of curves plotted this time at constant Poisson ratio (0.25) and shear modulus (30 GPa). In this figure, the confining pressure has been translated into depth using a constant density of 2900 kg.m$^{-3}$ to make the parametric study relevant to Earth. These curves outline that the confining pressure exerts a first order control on the amount of strain needed to reach steady state. However, even at an extreme depth, like 50 km, the maximum strain needed to achieve the phase of softening is close to 15%. Hence, structural softening is much more efficient as a strain softening mechanism than any existing mechanism based on mechanical wear. At depths more relevant for faulting (e.g. 12 km), the amount of accumulated shear strain is approximately halved. Given a fault of 5 cm thickness, such a strain corresponds to slip of 3.5 mm.

Both the confining pressure and shear modulus of rocks only appear as a proportionality constant in the incremental formulation for $\mathcal{F}$ (Eq. 34). It results that $\gamma_c$ increases linearly as the ratio $\frac{p_0}{G}$ increases. The elastic compressibility term, $\psi$, appears not only in the linear factor $(1 - \nu)$ in the incremental formulation of $\sigma^\text{elas}_\text{elas}^p$ (Eq. 64) but also as a non-linear term in the formula for $\gamma_\text{elas}^p$ (Eq. 65). Nevertheless, Fig. 5C shows that the value of finite strain in the shear band are proportional to $L_{\text{elas}}$ for a given shear band orientation ($\alpha_0$) and a set of plastic parameters ($\phi$ and $\psi$) according to

$$L_{\text{elas}} = -\frac{(1 - \nu)p_0}{G}.\quad (57)$$

This relationship implies that the non-linear compressibility term $1 - 2\nu$ which is embedded in Eq. (65) has very little impact on the characteristic strain. This is an important remark because this term is solely a result of the plane strain approximation made in this model.
**Effect of shear band orientation**

The shape of the stress-strain curves depends primarily on the orientation of the shear band. Fig. 6A represents the evolution of strength $\mathcal{A}$, tangent modulus $\Gamma$ and secant modulus $\Sigma$ versus non-dimensional shear strain within the band. Selected curves corresponding to the special shear band orientations described in the section on limit analysis are shown in the little insets. However, to view the whole range of admissible shear band orientations, we represent the all the curve by a surface which is colored as a function of non-dimensional shear strain $\gamma$ and shear band orientation $\alpha_0$. The inset are therefore cross section through this 3D surface. On these three figures, the right edge of the colored section represent the value $\gamma$. Firstly, it should be noted that, despite the total amount of stress rotation ($\alpha - \psi$) within the band, and hence the strength drop, increases with $\alpha_0$, the theoretical critical shear strain $\gamma_c$ decreases. $\gamma_c$ is particularly

---

*FIG. 6 – A : Surface plots of $\mathcal{A}$, $\Gamma$ and $\Sigma$ as a function of non-dimensional shear strain $\gamma$ and shear band orientation $\alpha_0$; B : Surface plots of $\Sigma$ as a function of the internal friction $\phi$ for two different shear band orientations $\alpha_A$ on the left and $\alpha_V$ on the right and two different values of dilation angle : $\psi = 10^\circ$ at the top and $\psi = 0^\circ$ at the bottom. In the middle, cross sections of the surface plots are plotted for typical rock friction coefficient of 0.6 and 0.85. FIGURE 6 – A : Carte des valeurs de $\mathcal{A}$, $\Gamma$ et $\Sigma$ en fonction de la déformation cisaillante $\gamma$ et de l’orientation de la bande de cisaillement $\alpha_0$ ; B : Carte des valeurs de $\Sigma$ en fonction de l’angle de friction interne $\phi$ pour deux orientations de bande de cisaillement $\alpha_A$ à gauche et $\alpha_V$ à droite ainsi que pour deux valeurs de l’angle de dilatation $\psi = 10^\circ$ en haut et $\psi = 0^\circ$ en bas. Au centre, courbes en coupe des cartes précédentes pour les coefficients de friction typique des roches de 0.6 et 0.85.*
large for shear band orientations, $\alpha_0$, within the range $[\alpha_{\text{eq}}, \alpha_0]$. Such shear bands also have the peculiarity that both their secant and tangent modulus decrease with increasing strain (see the insets curves on Fig. 6A). Therefore, we believe that within the range of possible shear band orientations, those possessing such characteristics have a limited chance to form within a MC-plasticity model unless some kinematic forcing is involved. Moreover, although the Arthur type shear bands ($\alpha_0 = \alpha_{\text{eq}}$) possess the largest initial rate of softening ($\Gamma_0$), the Coulomb type shear bands ($\alpha_0 = \alpha_c$) and their Vermeer type siblings ($\alpha_0 = \alpha_v$) have the smallest $\gamma_c$ and a larger stress drop. Coulomb and Vermeer type shear bands indeed have the same secant modulus, $\Sigma$, at the end of the softening phase, and although Coulomb shear band may present a higher softening rate locally $\Gamma$, the secant modulus of Vermeer shear bands is always smaller. Hence, we conclude that if there are no kinematic restrictions influencing the orientation of the shear band, the most favorable orientation is the Vermeer shear band.

Effects of plastic parameters

Friction does not influence the characteristic strain $\gamma_c$ as much as the orientation of shear band ($\alpha_0$). The characteristic strain $\gamma_c$ generally grows with friction. This growth is nearly linear for a shear band with an orientation lower than $\alpha_0$, or in which dilation is important (see Fig. 6b). However, shear bands with low dilation and orientation, i.e. $\alpha_0$ greater than $\alpha_{\text{eq}}$, exhibit a non linear relationship between the friction and $\gamma_c$. The curves show a clear maximum of $\gamma_c$ for frictions which are close to the value obtained for rocks in the laboratory, i.e. 35°. However, since friction clearly controls the amount of stress drop, it is clear that as friction drops, the secant modulus of the shear bands significantly drops. We did not represent the cross section for a friction coefficient of $\tan \phi = 0.2$ in Fig. 6b, because they were indistinguishable from the frame of the figure, i.e. the secant modulus for a material with a small friction ($\phi < 20°$) is one order of magnitude smaller than for a typical Byerlee [1978] friction parameter.

Dilation in shear bands ($\psi$) not only increases the steady state strength, but also the characteristic strain. For a reasonable dilatancy angle of 10°, the growth of $\gamma_c$ is about 30% on Fig. 6b. Both the reduction of the total stress drop $\mathcal{S}_{\text{tr}}$ and the characteristic strain $\gamma_c$ are well reflected in the value of the secant modulus $\Sigma$, which are itself halved when $\psi = 10°$. We found that the effect of the shear band related parameters ($\alpha_0$, $\phi$ and $\psi$) on the characteristic strain ($\gamma_c$) is quite non linear. However, from the parametric study an empirical rule may be defined. Such a rule is constructed by nothing but the following. Firstly, for a given orientation, we have seen that the characteristic strain increases with the residual strength ($\mathcal{S}_{\text{ss}}$, Eq. (48)) and decreases for large total stress drop ($\mathcal{S}_{\text{tr}}$, Eq. (49)), i.e. large $\phi$ and large $\alpha_0$. Secondly, $\alpha_0$ has a strong control on the amount of strain, and the dependence on $\alpha_0$ is non linear (Fig. 5A). Furthermore, the maximum value of $\alpha_0$ is close to $\psi$. Thirdly, taking a small value of $\psi$ and a large value of $\alpha_0$, compared to $\phi + \gamma$, leads to a smaller characteristic strain (Fig. 5B). Considering the above observations, we found that $L_c$ is a good empirical approximation of $\gamma_c$:

$$L_c = 4L_{\text{elas}}(\mathcal{S}_{\text{ss}} - 1 - \mathcal{S}_{\text{tr}})(1 - \tan(\alpha_0 - \psi)\tan(\alpha_0 - \alpha_{\text{eq}})).$$  \hspace{1cm} (58)

The measure of how well $L_c$ approximates $\gamma_c$ is shown in Fig. 7. Here, we computed over 1500 $\gamma_c$ using the semi analytical solution with $n = 5000$. The friction angle $\phi$ ranged from 5° to 40°, while the dilatation angle ranged from 0° to $\min[25°, \phi]$. All shear band orientations were explored, from the Roscoe angle to the Coulomb angle. For each set of parameters, we computed $L_c$ and found that this approximation allows to estimate the value of $\gamma_c$ within 7% error (Fig. 7).

![Figure 7 - Computed characteristic strain $\gamma_c$ vs approximation $L_c$ and residuals.](image)

**FIGURE 7** - Déformation caractéristique calculée $\gamma_c$ vs approximée $L_c$ et les résidus.

Application and Discussion

Applications

The analytic solution derived here represents an idealised and greatly simplified description of a shear band. To study more physically realistic shear banding scenarios, a full thermomechanical, elastoplastic numerical model has to be used. The semi analytical solution described in B can be used in a particular way to validate the implementation of the elastoplastic rheology required in such a numerical code. To that end, we provide in C, the matrix form of the shear banding problem as well as the derivation of the elastoplastic stiffness matrix $\mathbf{M}$ used in the paper. As input, this form of the solution requires only rheological parameters and as such exactly matches the input for the rheology routine in a fully coupled elastoplastic numerical model, provided both $\mathbf{D}$ (Eq. 21) and $\mathbf{M}$ (Eq. C) are known at the same point in space. By taking the $\mathbf{D}$ and $\mathbf{M}$ parameters from the elastoplastic code and comparing Eqs. (64) & (65) with the numerical approximation derived in the paper, the rheology procedure can be validated in isolation from the rest of the numerical scheme. Fig. 8A shows an example of using the semi analytic solution to validate a rheology procedure from a finite element elastoplastic code. Here we deliberately corrupted the numerical implementation given in C by introducing a factor two (a typical implementation mistake regarding the shear strain). In most case we observe that to reproduce the imposed stress drop, the corrupted implementation returns into the elastic domain. In other cases, the amount of strain is simply not correct.

The provided semi analytical solution can also be used to verify the accuracy of a code for a given number of time steps du-
ring the softening phase. Fig. 8B shows the results obtained with the simple algorithm given in C when 3, 5, 10 and 50 time steps are made and compare it to the analytical solution. One sees that increasing the number of steps from 10 to 50 is very expensive for the minimal gain in accuracy. On the other hand, with this algorithm one should not do less than 5 steps during the stress drop phase because the 3 step answer causes the elastic unloading of the shear band at the second time step (see how the cross is going backward on the strain axis). Therefore, we also provided the approximate function $L_x$, the results of these kinds of accuracy tests may be used to improve time stepping in numerical codes by fixing a maximum incremental strain during one time step. The strong dependence of $\gamma_c$ on the ratio of confining pressure versus shear modulus explains the lack of convergence most elasto-plastic codes and parcellary FLAC based methods exhibit when they are in the rigid limit; e.g. trying to reproduce sand box experiments as in Buitert et al. [2006].

**Brittle-plastic transition, orientation of shear bands and the MC-model**

The MC-model as well as other non associated flow rules are self-localizing flow rules. These models do not predict the occurrence of a single orientation of shear band, but rather a full range with three particular orientations. The first two, Vermeer and Arthur orientations are preferred because they represent the lowest $\gamma$ at steady state and highest initial rate of softening $\dot{\gamma}$. The last one, i.e. the Roscoe orientation, is special because it is the only orientation that can form when boundary conditions do not allow for large stress discontinuity across the fault. However this orientation do not produce any stress drop. If one wants to design a self localizing rule which would reach a single orientations as solution, a third parameter has to be introduced. This parameter denoted $h$ and set to 0 in Vermeer [1990] corresponds to a component of elastic hardening. If this parameter is set so that the rate of elastic hardening is of the order of the maximum rate of initial structural softening, the model will reach a unique solution. In that case, Arthur orientation will be chosen [Vermeer and Deborst, 1984 ]. However, the non uniqueness of possible shear band orientations obtained within the MC-model without this elastic hardening parameter actually introduces a possibility for complexity to exist within a single rheological model. In the following of this section, we discuss the aspects of natural and experimental observations which the MC-model can capture with only two independent parameters. In particular we focus on the brittle-plastic transition, re-occurrence of slip events, and orientation/distribution of microstructures present in fault zones. The strength drop corresponds to the capacity to elastically unload the medium surrounding a shear band. Hence, a shear band with no associated strength drop, cannot localize the strain and the deformation at the scale of the shear band is plastic. Oppositely, a shear band which allows for a large strength drop force its surrounding to return into the elastic regime so that at the scale of the shear band the deformation can be qualified as brittle. The MC-model does not lead to a unique solution in terms of shear band orientation. Based on those definitions of brittle and plastic, we can state that within the MC-model, the effective behavior of rocks passes a transition from brittle to plastic as the orientations of the neoformed shear band decreases towards the Roscoe orientation. This definition is consistent with laboratory observations which point out that, as the rocks crosses the brittle-plastic transition with increasing confining pressure, the orientation of the shear bands decreases towards 45° or less e.g. [Besuelle et al., 2000]. Vermeer [1990] observed the same transition of the orientation of the shear band in laboratory apparatus with increasing granulometry of the material and proposed that the shear bands formed with Roscoe angles at large grain size because the rubber jackets prohibits a large discontinuity of stress imposed by high angle shear bands on the side of the samples. We point out that since this discontinuity of stress is proportional to the confining pressure, larger samples should cross the brittle-plastic transition at higher confining pressure $p_0$, while materials with lower shear modulus, $G$ should cross it at lower confining pressure, $p_0$. The first case was described in Savage et al. [1996] where a change of apparatus provoked a systematic decrease of the effective friction $\mu_{eff}$. The second case tends to be confirmed by the fact that clays become plastic at a lower confining pressure, $p_0$ than sandstone. In both cases, laboratory experiments tend to confirm the validity of the model to describe rocks at laboratory scales. The characteristic strain $\gamma_c$ for full localization obtained through MC-model is very small. Hence, at larger strain, the neoformed faults start to rotate passively with slip and become less well or badly oriented again marking a new phase of fast localization. This cycles may explain complexities such as branching of the faults and the fact that in nature and experiments, the width of shear bands correlates with the amount of strain [Mair et al. 2000]. This model also gives insights on the neoformation of secondary shear within existing fault zone. Using this approach, Lecomte et al. [2011] have shown that the model predicts the neoformation of Riedel shears within the fault zone when they are plastically incompressible while Y-shear are favored when the faults dilate. The MC-model is widely used in geodynamics to simulate faulting in a pseudo-static regime. Kaus [2010] provides a review and a parametric study of the dip of the shear band in numerical experiments designed to be relevant for the Earth (i.e confining pressure is proportional to depth). The paper clearly outlines that thicker shear bands that need more finite displacement to reach full softening, tend to form at smaller angles to the maximum shear stress. In the case of visco-elasto-plastic simulations, these variations can be explained by the growing amount of strain needed to reach the cross-over between the curve representing the secant modulus of Arthur type shear bands and the one representing the Vermeer and Coulomb type shear bands (see Fig. 6 a).

In the extremely low resolution cases, the shear bands are shown to form with a Roscoe orientation. One can find a link between these simulations and experimental observations on larger grain
size shear bands [Vermeer, 1990]. For very low resolution, the shear bands are kinematically restricted by their interactions with the free surface of the model. Near this free surface, the stress rotation that is necessary for Coulomb and Arthur shear bands to form, is not allowed and therefore all shear bands tend to display a Roscoe orientation, e.g. [Buiter et al. 2006; Lemiale et al., 2008; Kaus 2010].

Natural fault behavior and the possible relevance of MC-model for dynamic instabilities

Earthquakes represent dynamic slip instabilities on a fault. The Drucker postulate [Drucker, 1966] states that slip must nucleate on a patch which behaves as a strain softening material, but Mandel [1966] has shown that this condition is only necessary but not sufficient to cause the collapse at large scale because kinematic restrictions can allow the stability of a body. However, Leroy and Ortiz [1990] have shown that it is possible to slip on a slip hardening fault (e.g. a badly oriented fault/slip barriers) in the dynamic regime as long as the first slip patch had a sufficient energy and therefore is sufficiently large. It is beyond the scope of this paper to discuss the dynamic propagation of slip instabilities on faults which is well described by rate and state friction given an initial slip patch exists e.g. [Kaneko et al. 2008]. However, it is useful to question the formation of the slip patch, i.e. before rate and state friction becomes predominant. The MC-model produces physically coherent strain softening models which are rather different from the classical strain softening model used in seismology (see Kanamori and Brodsky [2004] for review). The main difference lies in the fact that the MC-model secant modulus $\Sigma$ is not a constant with slip and varies with orientation. Its main advantage is that the critical strain, the rupture energy and the stress drop are physically defined. Moreover, as the characteristic strain depends on confining pressure $p_0$ in this model, the deeper the rupture nucleates, the larger is the amount of slip on the initial patch and therefore the rupture energy associated with the rupture. Hence, ruptures which nucleates in the deeper part of the brittle crust would have more chance to rupture slip barriers than those nucleating near the surface. This qualitative property of the MC-model is in accordance with the fact that large ruptures usually occur close to the brittle-ductile transition where the confining pressure is maximum. A second quality of the MC-model in modeling natural fault behavior is that it predicts that material with high Poisson ratio have higher secant ($\Sigma$) and tangent($\Gamma$) moduli and thus tend to be more unstable. In general, rocks have a Poisson ratio between 0.22 and 0.27. However, natural fault zone tends to be more elastically incompressible than their surrounding, either because they possesses a larger Poisson ratio (as high as 0.45, Faulkner et al. [2006]) or simply because their low permeability allows the build up of undrained pore pressure conditions during rapid slip. The MC-model is thus consistent with the fact that faults tends to slip several times on the same trend and that mature faults are weaker than their surrounding. It actually provides an alternative explanation of the frictional weakness of an evolved fault zone which is only verified in the laboratory when more than 50% of the bulk composition of the gouge consists of mineral clays [Numelin et al., 2007].

Conclusions

We derived stress drop and characteristic strains associated to the simplest rheology compatible with many major observations. Without any further increase in complexity of rheological model, this is sufficient to explain the strain localization, the orientation of the localized zones and their diversity. We deliberately excluded shear heating and tensile mode of fracturing in the model to concentrate on brittle fracturing in shear. Hence, the results obtained are only meant to be valid at seismogenic depth located between 3 and 20 km in the Earth. We have focused here on understanding the Mohr Coulomb rheology as a self localizing rheology in the crust. The MC-model without material softening is sufficient to localize shear strain in the brittle field without accounting for material softening. It naturally introduces softening of the system because its flow rule forces the principal axes of stress inside the shear band to rotate until the displacement becomes parallel to the fault. As a result, the effective friction exerted by the fault on the elastic media diminishes during the phase of rotation and reaches a steady states as soon as the fault is aligned to the transport direction (Roscoe orientation). This self-localizing flow rule does not predict the occurrence of a single orientation of shear bands but rather a full range with two peculiar orientations : Vermeer and Arthur orientations being preferred because they represent the weakest steady state and initial mechanism respectively. The spectrum of possible shear band orientations obtained with this model introduces a mechanism for complexity to exist within a single rheological model. The strain localization obtained through the MC-model represents only a few percent of elastic strain, which once translated into slip using a fault thickness of few cm, corresponds to a few mm to cm of slip in the brittle crust. This length scale is very small and corresponds to the seismic slip increments. Lecomte et al. [2012] have already applied it to the generation of multiplet on active low angle normal faults, it would be interesting to further test the prediction of this rheology within the dynamic regime in the future. The feedback between stress orientation, orientation of the plastic flow and the amount of elastic strain is non linear. This non linearity is sufficient to localize the stress without accounting for a reduction of the material peak friction as a result of mechanical wear as in e.g. Lavier et al. [2000], Frederiksen and Braun [2001] or Huismans et al. [2005]. The MC-model is consistent with numerous observations, both in the laboratory and on natural fault zone and thus constitutes a viable tool for those who are interested in modeling self localizing structures within a continuum approach. It was beyond the scope of this contribution to investigate the accuracy of different numerical strategies. Nevertheless, the semi-analytical solution provided here allows testing the accuracy of different schemes at a discrete point in a numerical mesh. Since we have quantified the characteristic strain, $\gamma_c$, the results of those fast accuracy tests can actually be used to develop adaptive time stepping strategies in numerical codes.

Acknowledgments : The author thanks Y.Y. Podladchikov for initiating the project, B. Huet for helping dealing with trigonometric function in maple and D.A. May for motivating her to submit. S.M. Schmalholz and M. Gerbault are thanks for their constructive reviews, especially Muriel for her patience double checking the equations.

SGF guest editors : Francoise Bergerat and Olivier Lacombe


A Linear system

Using Eqs. (30) and (31), to express our four unknowns \( x = [\lambda, \epsilon^0, \sigma^{in}, \sigma^{in}] \) in terms of \( \mathcal{J} \), the system can be written in the matrix form

\[
Ax = b,
\]

where

\[
A = \begin{bmatrix}
\frac{1-\nu}{2\nu} & 0 & 0 & -\frac{1}{2}(\sin \beta - \sin \psi) \\
0 & -1 & 0 & \frac{1}{2}(\sin \beta + \sin \psi) \\
\frac{1}{2}(\sin \phi - \sin \beta) & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
x = \begin{bmatrix}
\sigma^{in}_{\xi\xi} \\
\sigma^{in}_{\eta\eta} \\
\sigma^{in}_{\xi \eta} \\
\lambda
\end{bmatrix}, \quad b = \begin{bmatrix}
\frac{1-\nu}{2\nu} (1+\sin \alpha_0) \\
-\frac{1-\nu}{2\nu} (1-\sin \alpha_0) \\
\frac{1}{2} (1+\sin \alpha_0) (\sin \phi - \sin \beta) \\
0
\end{bmatrix},
\]

with \( Z \) given by

\[
Z = \frac{1}{2} (\cos(\alpha_0 - \beta) - \sin \phi).
\]

When \( \tilde{\beta} \) is within the bounds \( [\psi, \phi] \), solving the consistency condition (Eq. 31) for \( \lambda \) gives:

\[
\lambda = -\mathcal{J} \frac{p_0(1-\nu)(\cos(\alpha_0 - \beta) - \sin \phi)}{2G \det A}.
\]

The expression of stress rate can be obtained from the solution of Eq. 32 for \( \sigma^{in}_{\xi \eta} \) and from the continuity condition for the two other components of stress rate. It is expressed here as a deviation from the outside stress rate, reaching

\[
\dot{\sigma}^{in} = \dot{\sigma}^{out} + G \frac{\lambda}{1-\nu} \begin{bmatrix}
\sin \beta - \sin \psi & 0 \\
0 & 0
\end{bmatrix}.
\]

Note that all the terms containing the orientation of the shear band, \( \alpha_0 \), are contained within the outside stress rate term. Finally, the strain rate terms can also be written as a deviation from the elastic behavior:

\[
\dot{\epsilon}^{in} = \dot{\epsilon}^{out} + \lambda \begin{bmatrix}
0 \\
\frac{(1-2\nu)\sin \beta + \sin \psi}{\cos \beta}
\end{bmatrix}.
\]

B Semi analytical stress strain curves

We compute the displacement/shear strain accumulated at the steady state via a numerical approximation. In the following, the sign” will denote discrete variables. The total rotation of the principal stress is discretized in \( 2n \) equal increments which corresponds to \( 2n + 1 \) values of \( \tilde{\beta} \). We note that the elasto-plastic rheology is not strain-rate dependent, thus assuming that \( \tilde{\beta} \) is constant ensures that the time step \( \Delta t/2 \) is constant. We use a centered finite difference scheme in which the stress and finite strain are only evaluated at the \( n + 1 \) odd steps denoted \( i \), while the stress rate and strain rate are evaluated at the \( n \) even steps denoted \( i + 1/2 \). From \( \tilde{\beta} \), it is straight forward to compute \( \beta_{n,i} \) from Eq. (16) and \( \mathcal{J} \), with Eq. (15). From those two quantities, we can compute \( \mathcal{J}_{i+1/2} \) as

\[
\mathcal{J}_{i+1/2} = \frac{\mathcal{J}_{i+1} - \mathcal{J}_i}{\Delta t}.
\]

Using \( \mathcal{J}_{i+1/2} \) and \( \tilde{\beta}_{i+1/2} \), allows us to compute the stress rates and strain rates at the steps \( i + 1/2 \) using the analytical solution (Eqs. (64) & (65)). Integrating these expressions forward in time using Eq. (67), allows us to compute stress and finite strain according to

\[
\begin{align*}
\dot{\sigma}_{i+1} &= \dot{\sigma}_{i} + \frac{\mathcal{J}_i}{\Delta t}, \\
\dot{\epsilon}_{i+1} &= \dot{\epsilon}_{i} + \frac{\mathcal{J}_i}{\Delta t}.
\end{align*}
\]

C Example of numerical implementation

Here, we only describe a simple implementation of MCC-model in a FEM code. This implementation consist at computing by hand the elasto-plastic rheology stiffness matrix \( \mathbf{M} \). Given that when material yields, the stress rate may be written

\[
\dot{\mathbf{\sigma}} = \mathbf{D} \left( \dot{\mathbf{\epsilon}} - \lambda \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}} \right)
\]

and the consistency condition must be ensured, even in the case that some numerical errors persisted from the last time step

\[
\mathcal{F} = 0 \Rightarrow \frac{\mathcal{F}^{old}}{\Delta t} + \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}} \mathbf{D} \left( \dot{\mathbf{\epsilon}} - \lambda \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}} \right)
\]

one can rewrite the consistency condition as a function of stress rate

\[
\frac{\mathcal{F}^{old}}{\Delta t} = \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}} \frac{d}{d \Delta t} \mathbf{D} \left( \dot{\mathbf{\epsilon}} - \lambda \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}} \right)
\]

to solve for \( \lambda \) as a function of strain rate.

\[
\lambda = \frac{1}{d} \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}} \frac{d}{d \Delta t} \dot{\mathbf{\epsilon}} + \frac{\mathcal{F}^{old}}{d \Delta t}
\]

Re-injecting \( \lambda \) in the incremental stress update rule, the stress rate can now be decompose in a part that depends on strain rate and one that is independent of it.

\[
\dot{\mathbf{\sigma}} = \mathbf{M} \dot{\mathbf{\epsilon}} + \mathbf{R}^{M}
\]

in which

\[
\begin{align*}
\mathbf{M} &= \frac{1}{d} \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}} \frac{d}{d \Delta t} \mathbf{D}, \\
\mathbf{R}^{M} &= \frac{\mathcal{F}^{old}}{d \Delta t} \frac{d}{d \Delta t} \frac{\partial \mathcal{F}}{\partial \mathbf{\sigma}}
\end{align*}
\]
Both $\frac{\partial \sigma}{\partial \sigma}$ and $\frac{\partial \bar{F}}{\partial \sigma}$ are computed as a function of the "old" state of stress. Using these, a typical stress update in a numerical code take the form of:

$$\sigma_{\text{new}} = \sigma_{\text{old}} + R^M \Delta t + \Delta \dot{\epsilon}$$  \hspace{1cm} (74)

where the first two terms go to the right end side of the stress balance equation reaching a generic form like

$$B \Delta t MB V = B(\sigma_{\text{old}} + R^M \Delta t)$$  \hspace{1cm} (75)

with the operator

$$B = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$$  \hspace{1cm} (76)

MC rheology is very nonlinear, so before inserting it in a numerical code, it is useful to test the functions that assemble the stiffness matrix. To that purpose, we give here the matrix form of the problem for which we have derived a semi analytic solution.

$$\begin{bmatrix} 0 & 0 & 0 & -D_{11} & -D_{12} & -D_{13} \\ 0 & 0 & 0 & -D_{21} & -D_{22} & -D_{23} \\ 0 & 0 & 0 & -D_{31} & -D_{32} & -D_{33} \\ 1 & -M_{12} & -M_{13} & -M_{11} & 0 & 0 \\ 0 & -M_{22} & -M_{23} & -M_{21} & 0 & 0 \\ 0 & -M_{32} & -M_{33} & -M_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{\text{in}}^{\text{xx}} \\ \sigma_{\text{in}}^{\text{yy}} \\ \sigma_{\text{in}}^{\text{xy}} \\ \dot{\epsilon}_{\text{in}}^{\text{xx}} \\ \dot{\epsilon}_{\text{in}}^{\text{yy}} \\ \dot{\gamma}_{\text{in}}^{\text{xy}} \end{bmatrix} = \begin{bmatrix} -\frac{p_0}{2} (1 + \sin \alpha_0) \\ -\frac{p_0}{2} (1 - \sin \alpha_0) \\ \frac{p_0}{2} \cos \alpha_0 \\ R_1^M \\ R_2^M - \frac{p_0}{2} (1 - \sin \alpha_0) \\ R_3^M + \frac{p_0}{2} \cos \alpha_0 \end{bmatrix}$$  \hspace{1cm} (77)

To use this matrix form, one can compute the matrix $M$ and $R^M$ as he would do at the integration point of an FEM scheme and integrate on time as in B. The value obtained for $\bar{F}$ at each time step (here 2-50) can be compared with those obtain with the semi-analytical solution A using 2000 time steps to evaluate the accuracy of the implementation.